On the Dimensions of Cyclic Symmetry Classes of Tensors

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Abstract

The dimensions of the symmetry classes of tensors, associated with a certain cyclic subgroup of $\$_m$ which is generated by a product of disjoint cycles is explicitly given in terms of the generalized Ramanujan sum. These dimensions can also be expressed as the Euler φ-function and the Möbius function.

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1 Introduction

Let V be an n-dimensional vector space over the complex field $\mathbb{C}$. Let $\overset{m}{\otimes} V$ be the mth tensor power of V and write $\nu_1 \otimes \cdots \otimes \nu_m$ for the decomposable tensor product of the indicated vectors. To each permutation σ in $\$_m$ there corresponds a unique linear operator $P(\sigma) : \overset{m}{\otimes} V \to \overset{m}{\otimes} V$ determined by $P(\sigma)(\nu_1 \otimes \cdots \otimes \nu_m) = \nu_{\sigma^{-1}(1)} \otimes \cdots \otimes \nu_{\sigma^{-1}(m)}$. Let G be a subgroup of $\$_m$ and let $\mathrm{I}(G)$ be the set of all the irreducible complex characters of G. It follows from the orthogonality relations for characters that

$$\Big\{ T(G,\chi) : \overset{m}{\otimes} V \to \overset{m}{\otimes} V \Big| \; T(G,\chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma); \; \chi \in \mathrm{I}(G) \Big\}$$

is a set of annihilating idempotents which sum to the identity. The image of $\overset{m}{\otimes} V$ under the $T(G,\chi)$ is called the *symmetry class of tensors* associated with G and χ and is denoted by $V_\chi^m(G)$. It is well known that

$$\dim V_\chi^m(G) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) n^{c(\sigma)}, \tag{1}$$

where $c(\sigma)$ is the number of cycles, including cycles of length 1, in the disjoint cycle decomposition of σ (see [4]).

Several papers are devoted to calculating $\dim V_\chi^m(G)$ in a more closed form than (1). Cummings [2] in the case that G is a cyclic subgroup of $\$_m$ generated by a cycle of length m gives a formula for $\dim V_\chi^m(G)$ in terms of the Euler φ-function and considers the case that G is isomorphic to a direct product of cyclic groups as well. In [3] when G is the dihedral group of order $2m$ is considered and a formula is given when G is equal to the whole group $\$_m$ in [5] and [6]. In all cases $\dim V_\chi^m(G)$ involves certain functions of n.

In [7] there is a formula for calculating $\dim V_\chi^m(G)$ in the case that $G = \langle \pi_1 \rangle \cdots \langle \pi_p \rangle$, where π_is, $1 \le i \le p$, are disjoint cycles in $\$_m$. This formula involves the Euler φ-function and Möbius function and is a modification of the formula given in [2]. In this case if the order of π_i is m_i, $1 \le i \le p$, then $G \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_p}$. It is mentioned in [2] and [7] that if two groups are isomorphic, then the dimensions of the symmetry classes of tensors associated with them need not be equal. For example if $G = \langle (12) \rangle$ and $H = \langle (12)(34) \rangle$ are considered as subgroups of $\$_4$, then it is easy to calculate that $\dim V_{\chi_0}^4(G) = n^3(n+1)/2$ and $\dim V_{\chi_0}^4(H) = n^2(n^2+1)/2$ whereas $G \cong H \cong \mathbb{Z}_2$ and χ_0 is the identity character of $\mathbb{Z}_2$.

Now it is a natural question to consider the cyclic group $G = \langle \pi_1 \ldots \pi_p \rangle$ where the π_is, $1 \le i \le p$, are disjoint cycles and ask about the dimension of $V_\chi^m(G)$, where $\chi \in \mathrm{I}(G)$. In this case if the order of π_i, $1 \le i \le p$, is m_i, then $G \cong \mathbb{Z}_{[m_1,\ldots,m_p]}$, where $[m_1, \ldots, m_p]$ denotes the least common multiple of the integers $m_1, \ldots, m_p$. In this paper we obtain a formula for $\dim V_\chi^m(G)$ in the above case and this formula involves the generalized Ramanujan sum which itself involves the Euler φ-function and Möbius function. Therefore for the rest of this paper let $G < \$_m$ be of the form

$$G = \langle \pi_1 \ldots \pi_p \rangle,$$

where π_is, $1 \le i \le p$, are disjoint cycles in $\$_m$ of certain orders, say, $m_1, \ldots, m_p$, respectively. Since G is cyclic, therefore the irreducible characters of G are all linear and are of the form

$$\chi_h : \; G \to \mathbb{C}^*, \qquad \chi_h\left((\pi_1 \ldots \pi_p)^t\right) = \exp\left(\frac{2\pi i h t}{[m_1, \ldots, m_p]}\right), \\ 0 \le t \le [m_1, \ldots, m_p] - 1,$$

so

$$\mathrm{I}(G) = \Big\{ \chi_h : \; G \to \mathbb{C}^* \Big| \; 0 \le h \le [m_1, \ldots, m_p] - 1 \Big\},$$

where $[m_1, \ldots, m_p]$ denotes the least common multiple of the integers $m_1, \ldots, m_p$. The symbol $(m_1, \ldots, m_p)$ denotes the greatest common divisor of $m_1, \ldots, m_p$.

2 A Result About the Ramanujan Sum

The well known Ramanujan sum is

$$C_m(h) = \sum_{\substack{t\,=\,0 \\ (t,\,m)\,=\,1}}^{m-1} \exp\left(\frac{2\pi iht}{m}\right),$$

where m is a positive integer and h is a nonnegative integer. Ramanujan proved that (see [1])

$$C_m(h) = \frac{\varphi(m)\mu\big(m/(m,h)\big)}{\varphi\big(m/(m,h)\big)},$$

where φ is the Euler φ-function, i.e., $\varphi(1) = 1$; for $m > 1$, $\varphi(m) =$ the number of positive integers less than m and relatively prime to m, and μ is the Möbius function, i.e., $\mu(1) = 1$, $\mu(m) = 0$ if $p^2|m$ for some prime number p, and $\mu(m) = (-1)^r$ if $m = p_1 \ldots p_r$, where $p_1, \ldots, p_r$ are distinct prime numbers.

For our main result, we need to generalize the Ramanujan sum. It seems natural to us to generalize the Ramanujan sum as follows.

Definition 1 Let $m_1, \ldots, m_p$ be positive integers and let h be a nonnegative integer. Suppose $d_1|m_1, \ldots, d_p|m_p$. The *generalized Ramanujan sum* denoted by $S(h; m_1, \ldots, m_p; d_1, \ldots, d_p)$ is defined by

$$S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = \sum_{\substack{t\,=\,0 \\ (t,\,m_1)\,=\,d_1 \\ \vdots \\ (t,\,m_p)\,=\,d_p}}^{[m_1,\ldots,m_p]-1} \exp\left(\frac{2\pi iht}{[m_1, \ldots, m_p]}\right).$$

If the set $\big\{0 \le t \le [m_1, \ldots, m_p] - 1 |\ (t, m_i) = d_i;\ 1 \le i \le p\big\}$ is empty, then we define $S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = 0$.

Remark 1 It is obvious that $S(h;m;1) = C_m(h)$, and so the sum appearing in Definition 1 is a generalization of the Ramanujan sum.

In the following lemma we prove that the generalized Ramanujan sum defined in Definition 1 involves the Ramanujan sum.

Lemma 1 *Let $m_1, \ldots, m_p$ be positive integers and let h be a nonnegative integer. Suppose $d_1|m_1, \ldots, d_p|m_p$ and set $m'_i = m_i/d_i$, $M_i = m_1 \ldots m_p/m_i$, $M'_i = m'_1 \ldots m'_p/m'_i$, $D_i = d_1 \ldots d_p/d_i$ $(1 \le i \le p)$ and*

$$l = \frac{(M_1, \ldots, M_p)}{(M'_1, \ldots, M'_p)(D_1, \ldots, D_p)}.$$

Then we have

$$S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = \begin{cases} \frac{1}{l} C_{[m'_1, \ldots, m'_p]}(hl) & , \quad \text{if } \left(\frac{[d_1, \ldots, d_p]}{d_i}, m'_i\right)_{1 \le i \le p} = 1, \\ \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Proof. By Definition 1 and the fact that $\exp\left(2\pi iht/[m_1, \ldots, m_p]\right)$ is a periodic function of t with period $[m_1, \ldots, m_p]$ we have

$$\begin{aligned} S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) &= \sum_{\substack{t=0 \\ (t, m_1) = d_1 \\ \vdots \\ (t, m_p) = d_p}}^{[m_1, \ldots, m_p]-1} \exp\left(\frac{2\pi iht}{[m_1, \ldots, m_p]}\right) \\ &= \frac{1}{l} \sum_{\substack{t=0 \\ (t, m_1) = d_1 \\ \vdots \\ (t, m_p) = d_p}}^{l[m_1, \ldots, m_p]-1} \exp\left(\frac{2\pi iht}{[m_1, \ldots, m_p]}\right). \end{aligned}$$

Now letting $t = [d_1, \ldots, d_p]t'$, we obtain

$$
\begin{aligned}
S(h;m_1,\ldots,m_p;d_1,\ldots,d_p) &= \frac{1}{l} \sum_{\substack{t'=0 \\ ([d_1,\ldots,d_p]t',m_1)=d_1 \\ \vdots \\ ([d_1,\ldots,d_p]t',m_p)=d_p}}^{[m'_1,\ldots,m'_p]-1} \exp\left(\frac{2\pi i h l t'}{[m'_1,\ldots,m'_p]}\right) \\
&= \frac{1}{l} \sum_{\substack{t'=0 \\ (([d_1,\ldots,d_p]/d_1)\,t',m'_1)=1 \\ \vdots \\ (([d_1,\ldots,d_p]/d_p)\,t',m'_p)=1}}^{[m'_1,\ldots,m'_p]-1} \exp\left(\frac{2\pi i h l t'}{[m'_1,\ldots,m'_p]}\right).
\end{aligned}
$$

If $([d_1,\ldots,d_p]/d_i\,,m'_i)=1$ for all i, $1\le i\le p$, then the set of all the t's indexing the above summation is equal to the set of all the t's such that $0\le t'\le [m'_1,\ldots,m'_p]-1$ with conditions $(t',m'_i)=1$, $1\le i\le p$. And if there is an i for which $([d_1,\ldots,d_p]/d_i\,,m'_i)\neq 1$ then the above sum is zero and therefore we obtain:

$$
\begin{aligned}
S(h;m_1,\ldots,m_p;d_1,\ldots,d_p) &= \begin{cases} \dfrac{1}{l} \displaystyle\sum_{\substack{t'=0 \\ (t',m'_1)=1 \\ \vdots \\ (t',m'_p)=1}}^{[m'_1,\ldots,m'_p]-1} \exp\left(\frac{2\pi i h l t'}{[m'_1,\ldots,m'_p]}\right) &, \quad \text{if } \underset{1\le i\le p}{\left(\frac{[d_1,\ldots,d_p]}{d_i}, m'_i\right)=1}, \\ 0 &, \quad \text{otherwise,} \end{cases} \\
&= \begin{cases} \dfrac{1}{l} \displaystyle\sum_{\substack{t'=0 \\ (t',[m'_1,\ldots,m'_p])=1}}^{[m'_1,\ldots,m'_p]-1} \exp\left(\frac{2\pi i h l t'}{[m'_1,\ldots,m'_p]}\right) &, \quad \text{if } \underset{1\le i\le p}{\left(\frac{[d_1,\ldots,d_p]}{d_i}, m'_i\right)=1}, \\ 0 &, \quad \text{otherwise,} \end{cases} \\
&= \begin{cases} \dfrac{1}{l} C_{[m'_1,\ldots,m'_p]}(hl) &, \quad \text{if } \underset{1\le i\le p}{\left(\frac{[d_1,\ldots,d_p]}{d_i}, m'_i\right)=1}, \\ 0 &, \quad \text{otherwise.} \end{cases} \qquad \square
\end{aligned}
$$

In some special cases the generalized Ramanujan sum is given in the following examples:

Example 1 $S(0;m;d) = C_{m/d}(0) = \frac{\varphi(m/d)\mu\big((m/d)/(m/d,0)\big)}{\varphi\big((m/d)/(m/d,0)\big)} = \varphi(m/d).$

Example 2 If $(h,m) = 1$; we have

$$S(h;m;d) = C_{m/d}(h) = \frac{\varphi(m/d)\mu\big((m/d)/(m/d,h)\big)}{\varphi\big((m/d)/(m/d,h)\big)} = \mu(m/d).$$

3 The Dimensions of Some Symmetry Classes of Tensors

In this section, as we mentioned earlier, the group $G = \langle \pi_1 \dots \pi_p \rangle$ is considered, where the π_is, $1 \le i \le p$, are disjoint cycles in $\$_m$. Our aim is to calculate $\dim V_\chi^m(G)$, where $\chi \in \mathrm{I}(G)$, in terms of known functions. Our formula involves the generalized Ramanujan sum.

Theorem 1 *Let $G = \langle \pi_1 \dots \pi_p \rangle$, where the π_is, $1 \le i \le p$, are disjoint cycles in $\$_m$ of orders $m_1, \dots, m_p$, respectively, and let χ_h, $0 \le h \le [m_1, \dots, m_p] - 1$, be an irreducible complex character of G. Then*

$$\dim V_{\chi_h}^m(G) = \frac{n^{m-(m_1+\cdots+m_p)}}{[m_1, \dots, m_p]} \sum_{\substack{d_1 | m_1 \\ \vdots \\ d_p | m_p}} S(h; m_1, \dots, m_p; d_1, \dots, d_p) n^{d_1 + \cdots + d_p},$$

where $S(h; m_1, \dots, m_p; d_1, \dots, d_p)$ denotes the generalized Ramanujan sum.

Proof. According to (1) the dimension of $V_{\chi_h}^m(G)$ is

$$\frac{1}{[m_1, \dots, m_p]} \sum_{\sigma \in G} \chi_h(\sigma) n^{c(\sigma)}, \tag{2}$$

where $c(\sigma)$ denotes the number of cycles, including cycles of length 1, in the disjoint cycle decomposition of σ. But every $\sigma \in G$ is equal to $\sigma = (\pi_1 \dots \pi_p)^t$ for some t, $0 \le t \le [m_1, \dots, m_p] - 1$. Since $\pi_1, \dots, \pi_p$ are disjoint, so are $(\pi_1 \dots \pi_p)^t = \pi_1^t \dots \pi_p^t$. Appealing to [7] we can obtain

$$c(\pi_1^t \dots \pi_p^t) = c(\pi_1^t) + \cdots + c(\pi_p^t) + m - (m_1 + \cdots + m_p).$$

Note that if $(t, m_i) = d$, then π_i^t has d cycles of length m_i/d and therefore $c(\pi_i^t) = d = (t, m_i)$. So we have

$$c(\pi_1^t \dots \pi_p^t) = (t, m_1) + \cdots + (t, m_p) + m - (m_1 + \cdots + m_p).$$

Hence according to (2) we have

$$\begin{aligned}
\dim V_{\chi_h}^m(G) &= \frac{1}{[m_1, \dots, m_p]} \sum_{\sigma \in G} \chi_h(\sigma) n^{c(\sigma)} \\
&= \frac{1}{[m_1, \dots, m_p]} \sum_{t=0}^{[m_1,\dots,m_p]-1} \chi_h(\pi_1^t \dots \pi_p^t) n^{c(\pi_1^t \dots \pi_p^t)}
\end{aligned}$$

$$= \frac{1}{[m_1, \dots, m_p]} \sum_{t=0}^{[m_1,\dots,m_p]-1} \exp\left(\frac{2\pi i h t}{[m_1, \dots, m_p]}\right) n^{(t,m_1)+\cdots+(t,m_p)+m-(m_1+\cdots+m_p)}$$

$$= \frac{n^{m-(m_1+\cdots+m_p)}}{[m_1, \dots, m_p]} \sum_{t=0}^{[m_1,\dots,m_p]-1} \exp\left(\frac{2\pi i h t}{[m_1, \dots, m_p]}\right) n^{(t,m_1)+\cdots+(t,m_p)}.$$

Now letting $(t, m_i) = d_i$, $1 \leq i \leq p$, we obtain

$$\begin{aligned}
\dim V_{\chi_h}^m(G) &= \frac{n^{m-(m_1+\cdots+m_p)}}{[m_1, \dots, m_p]} \sum_{\substack{d_1 | m_1 \\ \vdots \\ d_p | m_p}} \left(\sum_{\substack{t=0 \\ (t,m_1)=d_1 \\ \vdots \\ (t,m_p)=d_p}}^{[m_1,\dots,m_p]-1} \exp\left(\frac{2\pi i h t}{[m_1, \dots, m_p]}\right) \right) n^{d_1+\cdots+d_p} \\
&= \frac{n^{m-(m_1+\cdots+m_p)}}{[m_1, \dots, m_p]} \sum_{\substack{d_1 | m_1 \\ \vdots \\ d_p | m_p}} S(h; m_1, \dots, m_p; d_1, \dots, d_p) n^{d_1+\cdots+d_p}. \quad \square
\end{aligned}$$

Using Theorem 1 we obtain Theorems 1 and 2 of [2] in the following corollaries.

Corollary 1 *If G is a cyclic subgroup of $\$_m$ generated by an m-cycle and χ is the identity character 1, then* $\dim V_\chi^m(G) = (1/m)\sum_{d|m}\varphi(m/d)n^d$.

Proof. Since $\chi = \chi_0$, by Example 1 and using Theorem 1 we obtain

$$\dim V_\chi^m(G) = \frac{n^{m-m}}{m}\sum_{d|m} S(0;m;d)n^d = \frac{1}{m}\sum_{d|m}\varphi(m/d)n^d. \quad \square$$

Remark 2 In Corollary 1, if $\dim V = n = 1$, then $\dim \overset{m}{\otimes} V = 1$, and so $\dim V_\chi^m(G) = 0$ or 1. So $(1/m)\sum_{d|m}\varphi(m/d) = 0$ or 1. But $(1/m)\sum_{d|m}\varphi(m/d) = 0$ is impossible, therefore $(1/m)\sum_{d|m}\varphi(m/d) = 1$ or $\sum_{d|m}\varphi(d) = m$, which is well known identity in number theory.

Corollary 2 *If G is a cyclic subgroup of $\$_m$ generated by an m-cycle and χ is a primitive linear character, then* $\dim V_\chi^m(G) = (1/m)\sum_{d|m}\mu(m/d)n^d$.

Proof. We know that a linear character of a cyclic subgroup of $\$_m$ is primitive if its value on a generator of the subgroup is a primitive mth root of unity, so $\chi = \chi_h$ where $(h,m) = 1$ and by Example 2 and using Theorem 1, we have

$$\dim V_\chi^m(G) = \frac{n^{m-m}}{m}\sum_{d|m} S(h;m;d)n^d = \frac{1}{m}\sum_{d|m}\mu(m/d)n^d. \quad \square$$

Example 3 Let $G = \langle(12)(34)(5678)\rangle$ be a subgroup of $\$_9$. Suppose χ is the identity character 1, i.e., $\chi = \chi_0$. Then by Theorem 1 we have

$$\dim V_\chi^9(G) = \frac{n^{9-(2+2+4)}}{4}\sum_{\substack{d_1|2\\ d_2|2\\ d_3|4}} S(0;2,2,4;d_1,d_2,d_3)n^{d_1+d_2+d_3}$$

$$= \frac{n}{4}\left[S(0;2,2,4;2,2,4)n^8 + S(0;2,2,4;2,2,2)n^6 + S(0;2,2,4;1,1,1)n^3\right]$$

$$= \frac{n}{4}[n^8 + n^6 + 2n^3].$$

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